

# Supplemental material for “Modeling the excess cell surface stored in a complex morphology of bleb-like protrusions”

## 1 The full set of coupled model equations

By postulating the linear momentum balance and incompressibility of the material system, the governing system of equations is (recall the description of the total free energy  $F$  in the main text):

$$\left\{ \begin{array}{l} \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma}, \\ \nabla \cdot \mathbf{v} = 0, \\ \partial_t \phi_i + \nabla \cdot (\mathbf{v} \phi_i) = \nabla \cdot (\lambda_i \nabla (\mu_i - \mu_2)), \quad i = 1, 3, \\ \partial_t \mathbf{p} + \mathbf{v} \cdot \nabla \mathbf{p} - \mathbf{W} \cdot \mathbf{p} = \nu \mathbf{D} \cdot \mathbf{p} + \lambda_p^{-1} \mathbf{h}, \end{array} \right. \quad (1)$$

where  $\mathbf{h} = -\frac{\delta F}{\delta \mathbf{p}}$  is known as the molecular field in the liquid crystal community [3], representing a torque generated by the Frank-Oseen elastic energy,  $\mu_i = \frac{\delta F}{\delta \phi_i}$  is the chemical potential with respect to  $\phi_i$ , given by

$$\begin{aligned} \mu_3 &= 3\sqrt{2}\gamma_{3s}\varepsilon(-\nabla^2\phi_3 + f'_s(\phi_3)) + \gamma_{123}\phi_1^2\phi_2^2\phi_3, \\ \mu_2 &= 3\sqrt{2}\gamma_{2s}\varepsilon(-\nabla^2\phi_2 + f'_s(\phi_2)) + \gamma_{123}\phi_1^2\phi_2\phi_3^2 + \phi_2\left(\frac{K}{2}(\nabla\mathbf{p})^2 + \frac{h_2}{4}|\mathbf{p}|^4 - \frac{h_1}{2}|\mathbf{p}|^2\right), \\ \mu_1 &= 3\sqrt{2}\gamma_{1s}\varepsilon(-\nabla^2\phi_1 + f'_s(\phi_1)) + \gamma_{123}\phi_1\phi_2^2\phi_3^2 \\ &\quad + 3\sqrt{2}\gamma_{1b}\left(\nabla^4\phi_1 - f'_b(\phi_1)\nabla^2\phi_1 - \nabla^2f_b(\phi_1) + f'_b(\phi_1)f_b(\phi_1)\right) - \alpha_1\nabla \cdot ((\mathbf{p} \cdot \nabla\phi_1)\mathbf{p}), \\ \mathbf{h} &= -K\nabla \cdot \left(\frac{\phi_2^2}{2}\nabla\mathbf{p}\right) + \frac{\phi_2^2}{2}(-h_1 + h_2|\mathbf{p}|^2)\mathbf{p} + \alpha_1(\mathbf{p} \cdot \nabla\phi_1)\nabla\phi_1, \end{aligned} \quad (2)$$

with  $f_s(\phi) = \frac{1}{\varepsilon^2}\phi^2(1-\phi)^2$  and  $f_b(\phi) = \frac{1}{\varepsilon^2}\phi(\phi-1)(2\phi-1 + \frac{\varepsilon}{\sqrt{2}}C_1)$ , where  $C_1$  is the spontaneous curvature functional defined in the main text and approximated from the phase field variable  $\phi_1$  by (6) below. Here  $\sigma$  is the total extra stress,  $\mathbf{W}_{\alpha\beta} = \frac{1}{2}(\partial_\beta\mathbf{v}_\alpha - \partial_\alpha\mathbf{v}_\beta)$  is the vorticity tensor,  $\mathbf{D}_{\alpha\beta} = \frac{1}{2}(\partial_\beta\mathbf{v}_\alpha + \partial_\alpha\mathbf{v}_\beta)$  is the rate of strain tensor,  $\nu$  is a geometric parameter for the nematic gel, and  $\lambda_p$  is a rotational relaxation time for the nematic director  $\mathbf{p}$ . Here the total extra stress tensor consists of three parts:

$$\sigma = \sigma^r + \sigma^d + \sigma^e, \quad (3)$$

where  $\sigma^r$  is the elastic stress corresponding to the motion of the nematic director  $\mathbf{p}$ ,  $\sigma^d$  is the viscous stress associated to the solvent in the system,  $\sigma^e$  is the Ericksen stress, the stress associated to the elastic interfacial force due to molecular convection. They are given specifically by the following:

$$\begin{aligned} \sigma^r &= -\frac{\nu}{2}(\mathbf{p}\mathbf{h} + \mathbf{h}\mathbf{p}) + \frac{1}{2}(\mathbf{p}\mathbf{h} - \mathbf{h}\mathbf{p}), \\ \sigma^d &= 2\eta\mathbf{D}, \\ \sigma_{\alpha\beta}^e &= (f - \sum_{i=1}^3 \phi_i\mu_i)\delta_{\alpha\beta} - \sum_{i=1}^3 \frac{\partial f}{\partial(\partial_\beta\phi_i)}\partial_\alpha\phi_i - \frac{\partial f}{\partial(\partial_\beta\mathbf{p}_\gamma)}\partial_\alpha\mathbf{p}_\gamma, \end{aligned} \quad (4)$$

where  $\eta$  is the volume-averaged viscosity,  $\eta = \sum_{i=1}^3 \phi_i \eta_i$ , with  $\eta_1$  the buffer viscosity,  $\eta_2$  the cortical viscosity and  $\eta_3$  the cytosol viscosity. The divergence of the Ericksen stress yields the interfacial forces  $\nabla \cdot \sigma^e = -(\nabla \mathbf{p}) \cdot \mathbf{h} - \sum_{i=1}^3 \phi_i \mu_i$ .

## 2 How to fit the spontaneous curvature

Given the TEM (2d) micrograph, we convert the experimental image into gray scale, where each pixel is represented by a number between 0 and 1 that represents the volume fraction of pure buffer, denoted by  $\phi_1$  (1 represents pure buffer, 0 represents cytosol and cortex). The plasma membrane is the level set  $\phi_1 = \frac{1}{2}$ . Then, the figure is interpolated into either 256\*256 pixels, or 128\*128 pixels, depending on the mesh size in the simulation. In a similar manner, for the 3D case, we use the data generated by the seed-and-grow model. The data is stored in either a 256<sup>3</sup> or 128<sup>3</sup> matrix, with each data point between 0 and 1, representing the volume fraction of pure buffer. All data are in Cartesian coordinates.

Denote the unit outward normal vector of the plasma membrane  $\{\phi_1 = \frac{1}{2}\}$  as  $\mathbf{n}$ , which is defined within the phase field context by

$$\mathbf{n} = -\frac{\nabla \phi_1}{|\nabla \phi_1|}. \quad (5)$$

Then the interfacial curvature can be expressed as a function of the phase field variable  $\phi_1$  via [15]

$$C_1 = -\nabla \cdot \mathbf{n} \approx \frac{1}{|\nabla \phi_1|} \left( \nabla^2 \phi_1 - \frac{\phi_1(1 - \phi_1)(1 - 2\phi_1)}{\varepsilon^2} \right). \quad (6)$$

## 3 Table of Model Parameters

All parameters are summarized in Table 1, including references that cite the order of magnitude for some, whereas the others are estimated.

Table 1: dimensional and dimensionless parameters

| Symbol                     | Description  | Nominal value       | Unit                  | Reference and Remarks                                     |
|----------------------------|--|---------------------|-----------------------|---|
| $d$                        | Characteristic length scale                                | $1 \times 10^{-5}$  | m                     | [14]  |
| $t_0$                      | Characteristic time scale                                  | 1                   | s                     | [14]  |
| $\rho$                     | Cell density   | $1.1 \times 10^3$   | kg/m <sup>3</sup>     | [5]   |
| $\eta_1, \eta_2, \eta_3$   | Averaged viscosity   | 10.0                | N · s/m <sup>2</sup>  | 10 <sup>-3</sup> – 1 [14] for cytosol, 100 [8] for cortex |
| $\gamma_{is}, i = 1, 2, 3$ | Surface tension for the interface                          | $5 \times 10^{-5}$  | N/m                   | [10]  |
| $\gamma_{1b}$              | Bending rigidity of cell membrane                          | $1 \times 10^{-18}$ | N · m                 | [13]  |
| $\varepsilon$              | Thickness of the interface                                 | $5 \times 10^{-7}$  | m                     | model parameter   |
| $\lambda_1, \lambda_3$     | Motility parameter   | $1 \times 10^{-8}$  | m <sup>3</sup> · s/kg | model parameter   |
| $\lambda_p$                | Time relaxation for the nematic director $\mathbf{p}$      | $1 \times 10^3$     | s                     | [4]   |
| $s_0$                      | Excess surface area ratio                                  | 3                   |                       | experiment measured                                       |
| $\lambda_S$                | Lagrange multiplier for excess surface area constraint     | $2 \times 10^5$     | N/m <sup>3</sup>      | model parameter   |
| $K$                        | elastic strength for cell cortex (Frank elastic constants) | $10^{-11}$          | N                     | [16, 9]   |
| $h_1, h_2$                 | Landau-De Gennes nematic potential parameters              | $2 \times 10^2$     | N/m <sup>2</sup>      | model parameter   |
| $\alpha_1$                 | Parallel anchoring strength                                | $10^{-11}$          | N                     | assume the same with $K$                                  |
| $\nu$                      | Nematic director tumbling parameter                        | 1.2                 |                       | rod-like flow-aligning regime                             |

## 4 Numerical Schemes

We now present the semi-discrete numerical scheme as follows.

Given the initial conditions  $(\mathbf{p}^0, \phi_1^0, \phi_2^0, \phi_3^0, \mathbf{v}^0 = 0, p^0 = 0)$ , having computed  $(\mathbf{p}^n, \phi_1^n, \phi_2^n, \phi_3^n, \mathbf{v}^n, p^n)$  for  $n \geq 0$ , we compute  $(\mathbf{p}^{n+1}, \phi_1^{n+1}, \phi_2^{n+1}, \phi_3^{n+1}, \mathbf{v}^{n+1}, p^{n+1})$  in the following sequence.

1. Step 1: update  $\mathbf{p}^{n+1}$ :

$$\left\{ \begin{array}{l} \frac{\mathbf{p}^{n+1} - \mathbf{p}^n}{\delta t} + \mathbf{v}^n \cdot \nabla \mathbf{p}^n - \frac{1}{2}(\mathbf{v}^n - (\mathbf{v}^n)^T) \cdot \mathbf{p}^n - \frac{\nu}{2}(\mathbf{v}^n + (\mathbf{v}^n)^T) \cdot \mathbf{p}^n = -\frac{1}{\lambda_p} \mathbf{h}^{n+1}, \\ \mathbf{h}^{n+1} = -K \nabla \cdot \left( \left( \frac{(\phi_2^n)^2}{2} \nabla \mathbf{p}^{n+1} \right) + \frac{(\phi_2^n)^2}{2} (-h_1 + h_2 \|\mathbf{p}^n\|^2) \mathbf{p}^{n+1}, \right. \\ \left. \frac{\partial \mathbf{p}^{n+1}}{\partial \mathbf{n}} \Big|_{\partial \Omega} = 0, \right. \end{array} \right. \quad (7)$$

2. Step 2: update  $(\phi_1^{n+1}, \phi_3^{n+1})$ :

$$\left\{ \begin{array}{l} \frac{\phi_i^{n+1} - \phi_i^n}{\delta t} + \nabla \cdot (\mathbf{v}^n \phi_i^n) = \nabla \cdot (\lambda_i \nabla (\mu_i^{n+1} - \mu_i^n)), \\ \mu_3^{n+1} = C_3 (\phi_3^{n+1} - \phi_3^n) + 3\sqrt{2} \gamma_{3s} \varepsilon (-\nabla^2 \phi_3^{n+1} + f'_s(\phi_3^n)) + \gamma_{123} (\phi_1^2 \phi_2^2 \phi_3)^n, \\ \mu_2^n = 3\sqrt{2} \gamma_{2s} \varepsilon (-\nabla^2 \phi_2^{n+1} + f'_s(\phi_2^n)) + \gamma_{123} (\phi_1^2 \phi_2 \phi_3^2)^n \\ + \phi_2^n \left( \frac{K}{2} (\nabla \mathbf{p}^{n+1})^2 + \frac{h_2}{4} \|\mathbf{p}^{n+1}\|^4 - \frac{h_1}{2} \|\mathbf{p}^{n+1}\|^2 \right), \\ \mu_1^{n+1} = C_2 (\phi_1^{n+1} - \phi_1^n) + 3\sqrt{2} \gamma_{1s} \varepsilon (-\nabla^2 \phi_1^{n+1} + f'_s(\phi_1^n)) + \gamma_{123} (\phi_1 \phi_2^2 \phi_3^2)^n + 3\sqrt{2} \gamma_{1b} \left( \right. \\ \left. \nabla^4 \phi_1^{n+1} - f'_b(\phi_1^n) \nabla^2 \phi_1^n - \nabla^2 f_b(\phi_1^n) + f'_b(\phi_1^n) f_b(\phi_1^n) \right) - \alpha_1 \nabla \cdot ((\mathbf{p}^{n+1} \cdot \nabla \phi_1^n) \mathbf{p}^{n+1}), \\ \frac{\partial \phi_i^{n+1}}{\partial \mathbf{n}} \Big|_{\partial \Omega} = 0, \frac{\partial \nabla^2 \phi_i^{n+1}}{\partial \mathbf{n}} \Big|_{\partial \Omega} = 0, \quad \frac{\partial \mu_i^{n+1}}{\partial \mathbf{n}} \Big|_{\partial \Omega} = 0, \end{array} \right. \quad (8)$$

3. Step 3: update  $\phi_2^{n+1}$ :

$$\phi_2^{n+1} = 1 - \phi_1^{n+1} - \phi_3^{n+1}. \quad (9)$$

4. Step 4: update  $(\mathbf{v}^{n+1}, p^{n+1})$ :

$$\left\{ \begin{array}{l} \rho \frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla) \tilde{\mathbf{v}}^{n+1} = \eta \nabla \cdot (\tilde{\mathbf{v}}^{n+1} + (\tilde{\mathbf{v}}^{n+1})^T) - \nabla p^n - \phi^n \nabla \mu^{n+1} - \mathbf{h}^{n+1} \nabla \mathbf{p}^n \\ + \nabla \cdot \left( -\frac{a}{2} (\mathbf{p}^n \mathbf{h}^{n+1} + \mathbf{h}^{n+1} \mathbf{p}^n) + \frac{1}{2} (\mathbf{p}^n \mathbf{h}^{n+1} - \mathbf{h}^{n+1} \mathbf{p}^n) + \zeta \phi_2^{n+1} \phi_3^{n+1} \mathbf{p}^{n+1} \mathbf{p}^{n+1} \right), \\ \tilde{\mathbf{v}}^{n+1} \Big|_{\partial \Omega} = 0. \end{array} \right. \quad (10)$$

$$\left\{ \begin{array}{l} \frac{\mathbf{v}^{n+1} - \tilde{\mathbf{v}}^{n+1}}{\delta t} = -\nabla (p^{n+1} - p^n), \\ \nabla \cdot \mathbf{v}^{n+1} = 0, \quad \mathbf{v}^{n+1} \Big|_{\partial \Omega} = 0. \end{array} \right. \quad (11)$$

In the above,  $C_2$  and  $C_3$  are numerical stabilizing parameters [11]. The above scheme is constructed by combining several effective approaches in the approximation of Cahn-Hilliard equations [11], Navier-Stokes equations [6] and phase-field models [12, 2].

The numerical scheme is further discretized in space by central finite differences, and implemented on graphic processing units (GPUs) using the CUDA interface. The Nvidia CUFFT, Thrust [7], as well as CUSP [1] have been used to solve the linearized systems. The resultant solver is tested in both time and space to ensure it is convergent and attains first order accuracy.

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